

THE TAUTOLOGICAL RING OF $M_{1,n}^{ct}$

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Introduction.

Let $M_{g,n}^{ct}$ be the moduli space of stable n -pointed genus g curves of compact type and denote by $R^*(M_{g,n}^{ct})$ its tautological ring. Here, we study this ring in genus one. It is known that the tautological ring $R^*(M_{1,n}^{ct})$ is additively generated by boundary cycles, and it is the subalgebra of the Chow ring $A^*(M_{1,n}^{ct})$ (taken with \mathbb{Q} -coefficients throughout) of $M_{1,n}^{ct}$ generated by divisor classes D_I , for $I \subset \{1, \dots, n\}$ with $|I| > 1$. Recall that a boundary cycle of $M_{1,n}^{ct}$ parameterizes stable curves whose dual graphs are trees, and D_I is associated to those with one edge, for which I is the marking set on the genus zero component. We study this ring to understand the space of relations among the generators. In particular, we prove that the tautological ring is Gorenstein.

We begin this note by recalling the definitions and known facts about the tautological algebras as well as the conjectural structure of them.

In the second section we consider a fixed pointed elliptic curve $(C; O)$, and we describe the reduced fiber of the projection $M_{1,n}^{ct} \rightarrow M_{1,1}^{ct}$ over $[(C; O)] \in M_{1,1}^{ct}$, which is denoted by \overline{U}_{n-1} , as a sequence of blow-ups of C^{n-1} . As a result, we get a map

$$F : \overline{U}_{n-1} \rightarrow M_{1,n}^{ct}.$$

There is a description of the Chow ring $A^*(\overline{U}_{n-1})$ of \overline{U}_{n-1} in the third section.

Then we define the tautological ring $R^*(C^n)$ of C^n as a subring of its Chow ring $A^*(C^n)$. We give a description of the pairing

$$R^d(C^n) \times R^{n-d}(C^n) \rightarrow \mathbb{Q}$$

for $0 \leq d \leq n$. In particular, we will see that this pairing is perfect.

The fifth section starts with the definition of the tautological ring $R^*(\overline{U}_{n-1})$ of \overline{U}_{n-1} . It is defined to be the subalgebra of its Chow ring generated by the tautological classes in $R^*(C^{n-1})$ and the classes of proper transforms of the exceptional divisors introduced in the construction of \overline{U}_{n-1} . The study of the pairing

$$R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1}) \rightarrow \mathbb{Q},$$

for $0 \leq d \leq n-1$, shows that it is perfect as well.

In the last section we study the fibers of the map $F : \overline{U}_{n-1} \rightarrow M_{1,n}^{ct}$, and we will see that the images of the tautological classes in $M_{1,n}^{ct}$ under the induced pull-back

$$F^* : A^*(M_{1,n}^{ct}) \rightarrow A^*(\overline{U}_{n-1})$$

are elements of the tautological ring $R^*(\overline{U}_{n-1})$ of \overline{U}_{n-1} and hence, it induces a map

$$R^*(M_{1,n}^{ct}) \rightarrow R^*(\overline{U}_{n-1}),$$

which is denoted by the same letter F^* , by abuse of notation. Then, we will see that F^* induces an isomorphism between the tautological rings involved. This gives a description of the ring $R^*(M_{1,n}^{ct})$ in terms of the generators D_I 's and the space of relations. In particular, from the proven result for $R^*(\overline{U}_{n-1})$, we conclude that $R^*(M_{1,n}^{ct})$ is a Gorenstein ring.

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1. Review of known facts and conjectures about the tautological ring $R^*(M_{g,n}^{ct})$

Let $\overline{M}_{g,n}$ be the moduli space of stable curves of genus g with n marked points. In [FP3] the system of tautological rings is defined to be the set of smallest \mathbb{Q} -subalgebras of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}),$$

satisfying the following two properties:

- The system is closed under push-forward via all maps forgetting markings:

$$\pi_* : R^*(\overline{M}_{g,n}) \rightarrow R^*(\overline{M}_{g,n-1}).$$

- The system is closed under push-forward via all gluing maps:

$$\iota_* : R^*(\overline{M}_{g_1, n_1 \cup \{*\}}) \otimes R^*(\overline{M}_{g_2, n_2 \cup \{\bullet\}}) \rightarrow R^*(\overline{M}_{g_1+g_2, n_1+n_2}),$$

$$\iota_* : R^*(\overline{M}_{g, n \cup \{*, \bullet\}}) \rightarrow R^*(\overline{M}_{g+1, n}),$$

with attachments along the markings $*$ and \bullet .

The standard ψ , κ and λ classes in $A^*(\overline{M}_{g,n})$ all lie in the tautological ring (see below for definition). The quotient $R^*(M_{g,n})$ of the tautological ring is defined as the restriction to the open subset $M_{g,n}$. In [F1] it was conjectured that the tautological ring $R^*(M_g)$ is a Gorenstein algebra with socle in degree $g-2$. It was raised as a question in [HL] whether the tautological ring of $\overline{M}_{g,n}$ satisfy Poincare duality and has the Lefschetz property with respect to κ_1 , which was known to be ample [C]. In [F2] the following conjecture about the tautological ring $R^*(\overline{M}_{g,n})$ is stated:

Conjecture 1.1. $R^*(\overline{M}_{g,n})$ is Gorenstein with socle in degree $3g-3+n$.

We now define the moduli space $M_{g,n}^{ct}$ and its tautological ring. To every n -pointed stable curve $(C; x_1, \dots, x_n)$ there is an associated dual graph. Its vertices correspond to the irreducible components of C and edges correspond to intersection of components. Note that self intersection is allowed. The curve C is of compact type if its dual graph is a tree, or equivalently, the Jacobian of C is an abelian variety. The moduli space $M_{g,n}^{ct}$ parametrizes stable n -pointed curves of genus g of compact type. One can also define $M_{g,n}^{ct}$ as the complement of the boundary divisor Δ_{irr} of irreducible singular curves and their degenerations.

The tautological ring, $R^*(M_{g,n}^{ct}) \subset A^*(M_{g,n}^{ct})$, for the moduli space $M_{g,n}^{ct}$, is defined to be the image of $R^*(\overline{M}_{g,n})$ via the natural map,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}) \rightarrow A^*(M_{g,n}^{ct}).$$

The quotient ring $R^*(M_{g,n}^{ct})$ admits a canonical non-trivial linear evaluation ϵ to \mathbb{Q} obtained by integration involving the λ_g class, the Euler class of the Hodge bundle.

Recall that the Hodge bundle \mathbb{E} is the locally free \mathcal{O} -sheaf of rank g defined by $\mathbb{E} = \pi_* \omega$, where $\pi : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$ forgets the marking $n+1$ and ω denotes its relative dualizing sheaf. The fiber of \mathbb{E} over a moduli point $[(C; x_1, \dots, x_n)]$ is the g -dimensional vector space $H^0(C, \omega_C)$. The class λ_i is defined to be the i^{th} Chern class $c_i(\mathbb{E})$ of the Hodge bundle. The class κ_i is defined to be the push-forward $\pi_*(K^{i+1})$, where $K = c_1(\omega)$. The class ψ_i is the pull back $\sigma_i^*(K)$ of K along $\sigma_i : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n+1}$, where $\sigma_1, \dots, \sigma_n$ are the natural sections of the map π . It is the first Chern class of the bundle on the moduli space whose fiber at the moduli point $[(C; x_1, \dots, x_n)]$ is the cotangent space to C at the i^{th} marking.

The class λ_g vanishes when restricted to the complement Δ_{irr} . This gives rise to an evaluation ϵ on $A^*(M_{g,n}^{ct})$:

$$\xi \mapsto \epsilon(\xi) = \int_{\overline{M}_{g,n}} \xi \cdot \lambda_g.$$

The non-triviality of the ϵ evaluation is proven by explicit integral computations. The following formula for λ_g integrals is proven in [FP2]:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \lambda_g = \binom{2g-3+n}{\alpha_1, \dots, \alpha_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g.$$

The integrals on the right side are evaluated in terms of the Bernoulli numbers:

$$\int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$

This proves the non-triviality of the evaluation since B_{2g} doesn't vanish.

It is proven in [GV] that $R^*(M_{g,n}^{ct})$ vanishes in degrees $> 2g - 3 + n$ and is 1-dimensional in degree $2g - 3 + n$. It was speculated in [FP1] that $R^*(M_g^{ct})$ is a Gorenstein algebra with socle in codimension $2g - 3$. The following conjecture is stated in [F2]:

Conjecture 1.2. $R^*(M_{g,n}^{ct})$ is Gorenstein with socle in degree $2g - 3 + n$.

A compactly supported version of the tautological algebra is defined in [HL]. The algebra $R_c^*(M_{g,n})$ is defined to be the set of elements in $R^*(\overline{M}_{g,n})$ that restrict trivially to the Deligne-Mumford boundary. This is a graded ideal in $R^*(\overline{M}_{g,n})$ and the intersection product defines a map

$$R^*(M_{g,n}) \times R_c^*(M_{g,n}) \rightarrow R_c^*(M_{g,n})$$

that makes $R_c^*(M_{g,n})$ a $R^*(M_{g,n})$ -module. In [HL] they formulated the following conjecture for the case $n = 0$:

Conjecture 1.3. (A) *The intersection pairings*

$$R^k(M_g) \times R_c^{3g-3-k}(M_g) \rightarrow R_c^{3g-3}(M_g) \cong \mathbb{Q}$$

are perfect for $k \geq 0$.

(B) *In addition to (A), $R_c^*(M_g)$ is a free $R^*(M_g)$ -module of rank one.*

In a similar fashion one defines $R_c^*(M_{g,n}^{ct})$ as the set of elements in $R^*(\overline{M}_{g,n})$ that pull back to zero via the standard map $\overline{M}_{g-1,n+2} \rightarrow \overline{M}_{g,n}$ onto Δ_{irr} . The analogue of the conjectures above for the spaces $M_{g,n}^{ct}$ instead of M_g and its relation with the conjecture 1.2 is discussed in [F3]. First consider the analogue of the conjectures 1.3 as follows:

Conjecture 1.4. (C) *The intersection pairings*

$$R^k(M_{g,n}^{ct}) \times R_c^{3g-3+n-k}(M_{g,n}^{ct}) \rightarrow R_c^{3g-3+n}(M_{g,n}^{ct}) \cong \mathbb{Q}$$

are perfect for $k \geq 0$.

(D) *In addition to C, $R_c^*(M_{g,n}^{ct})$ is a free $R^*(M_{g,n}^{ct})$ -module of rank one.*

In [F3] it is proven that for a given (g, n) , the statement D in 1.4 follows if the statements 1.1 and 1.2 hold. On the other hand, for such (g, n) the statements 1.2 and C in 1.4 follow from D in 1.4. It is also proven that a counterexample to the conjecture 1.1 leads to a disproof of the conjecture C in 1.4.

In this note we consider the case $g = 1$ and prove that the conjecture 1.2 is true in this case.

2. The space \overline{U}_{n-1}

Let C be a fixed elliptic curve and choose a point $O \in C$ as its origin. For a given natural number $n \in \mathbb{N}$, the space U_{n-1} is defined to be the open subset

$$\{(x_1, \dots, x_{n-1}) \in C^{n-1} : x_i \neq O \text{ and } x_i \neq x_j \text{ for } i \neq j\}$$

of C^{n-1} . The projection $\pi : U_{n-1} \times C \rightarrow U_{n-1}$ admits n disjoint sections with smooth fibers and defines a map

$$F : U_{n-1} \rightarrow M_{1,n},$$

where $M_{1,n}$ denotes the moduli space of smooth n -pointed curves of genus one. The map F sends the point $P = (x_1, \dots, x_{n-1})$ of U_{n-1} to the class of the pointed curve $(C; x_1, \dots, x_{n-1}, O)$.

For a subset I of $\{1, \dots, n\}$, let $X_I \subset C^{n-1}$ be the $|I|$ -dimensional subvariety defined by

$$\begin{cases} x_i = x_j & \text{for } i, j \in \{1, \dots, n\} - I \quad \text{if } n \in I \\ x_i = O & \text{for } i \in \{1, \dots, n-1\} - I \quad \text{if } n \notin I. \end{cases}$$

The space \overline{U}_{n-1} is constructed from C^{n-1} by the following sequence of blow-ups:

At step zero blow-up C^{n-1} at the point X_0 , and at the k^{th} step, for $1 \leq k \leq n-3$, blow-up the space obtained in the previous step along the regularly embedded union of the proper transforms of the subvarieties X_I , where $|I| = k$.

The space \overline{U}_{n-1} contains U_{n-1} as an open dense subset. There exists a family of stable curves of genus one of compact type over \overline{U}_{n-1} , whose total space is isomorphic to \overline{U}_n . The resulting family is denoted by $\pi : \overline{U}_n \rightarrow \overline{U}_{n-1}$ by abuse of notation. Since $\pi^{-1}(U_{n-1})$ is isomorphic to the product $U_{n-1} \times C$, on which π is projection onto the first factor, and this coincides with the former definition of π given above, there is no danger of confusion. The map π admits n disjoint sections in the smooth locus of the fibers, and defines a morphism

$$F : \overline{U}_{n-1} \rightarrow M_{1,n}^{\text{ct}}.$$

The morphism F sends a geometric point $P \in \overline{U}_{n-1}$ to the moduli point of the pointed curve $(\pi^{-1}(P); x_1, \dots, x_n)$, where the x_i 's are the n distinct points on the fiber $\pi^{-1}(P)$ obtained by intersecting the fiber $\pi^{-1}(P)$ with the n disjoint sections of π .

3. The Chow ring $A^*(\overline{U}_{n-1})$

In this section we recall some facts about the intersection ring of the blow-up \tilde{Y} of the smooth variety Y along a smooth irreducible subvariety Z from [FM]. When the restriction map from $A^*(Y)$ to $A^*(Z)$ is surjective, S. Keel has shown in [K] that the computations become simpler. We denote the kernel of the restriction map by $J_{Z/Y}$ so that

$$A^*(Z) = \frac{A^*(Y)}{J_{Z/Y}}.$$

Define a Chern polynomial for $Z \subset Y$, denoted by $P_{Z/Y}(t)$, to be a polynomial

$$P_{Z/Y}(t) = t^d + a_1 t^{d-1} + \dots + a_{d-1} t + a_d \in A^*(Y)[t],$$

where d is the codimension of Z in Y and $a_i \in A^i(Y)$ is a class whose restriction in $A^i(Z)$ is $c_i(N_{Z/Y})$, where $N_{Z/Y}$ is the normal bundle of Z in Y . We also require that $a_d = [Z]$, while the other classes a_i , for $0 < i < d$, are determined only modulo $J_{Z/Y}$.

Let us verify the surjectivity of the restriction map from $A^*(Y)$ to $A^*(Z)$ in our case, when $Y = C^{n-1}$ and $Z = X_I$, for a subset I of the set $\{1, \dots, n\}$. First assume that n doesn't belong to the set I . Denote by $i_I : X_I \rightarrow C^{n-1}$ the inclusion map and by $\pi : C^{n-1} \rightarrow X_I$ the canonical projection. From the equality $\pi \circ i_I = id_{X_I}$ we conclude that the restriction

map i_I^* is surjective. It also follows that the push-forward map $(i_I)_*$ is injective. This will be used in 5.1. The case $n \in I$ is treated in a similar manner. In this case there is not a canonical projection $\pi : C^{n-1} \rightarrow X_I$, and one has to make a choice.

The following lemma can be used to compute $P_{Z/Y}$ when the subvariety Z is a transversal intersection of divisor classes:

Lemma 3.1. (a) If $Z = D$ is a divisor, then $P_{D/Y}(t) = t + D$.

(b) If $V \subset Y$ and $W \subset Y$ are subvarieties meeting transversally in a variety Z , and V and W have Chern polynomials $P_{V/Y}(t)$ and $P_{W/Y}(t)$, then Z has a Chern polynomial

$$P_{Z/Y}(t) = P_{V/Y}(t) \cdot P_{W/Y}(t).$$

In addition the restriction from $A^*(Y)[t]$ to $A^*(V)[t]$ maps $P_{W/Y}(t)$ to a Chern polynomial $P_{Z/V}(t)$ for $Z \subset V$.

Proof. This is Lemma 5.1 in [FM]. \square

We identify $A^*(Y)$ as a subring of $A^*(\tilde{Y})$ by means of the map $\pi^* : A^*(Y) \rightarrow A^*(\tilde{Y})$, where $\pi : \tilde{Y} \rightarrow Y$ is the birational morphism. Let $E \subset \tilde{Y}$ be the exceptional divisor. The formula of Keel is as follows:

Lemma 3.2. With the above assumptions and notations, the Chow ring $A^*(\tilde{Y})$ is given by

$$A^*(\tilde{Y}) = \frac{A^*(Y)[E]}{(J_{Z/Y} \cdot E, P_{Z/Y}(-E))}.$$

Proof. This is Lemma 5.3 in [FM]. \square

The next lemma relates a Chern polynomial $P_{\tilde{V}/\tilde{Y}}(t)$ of the proper transform \tilde{V} of a subvariety $V \subset Y$ to $P_{V/Y}(t)$:

Lemma 3.3. Let V be a subvariety of Y not contained in Z and let $\tilde{V} \subset \tilde{Y}$ be its proper transform. Suppose that $P_{V/Y}(t)$ is a Chern polynomial for V .

- (1) If V meets Z transversally, then $P_{V/Y}(t)$ is a Chern polynomial for \tilde{V} in \tilde{Y} .
- (2) If V contains Z , then $P_{V/Y}(t - E)$ is a Chern polynomial for $\tilde{V} \subset \tilde{Y}$.

Proof. This is Lemma 5.2 in [FM]. \square

We also need the following lemmas to relate the ideal $J_{\tilde{V}/\tilde{Y}}$ to the ideal $J_{V/Y}$ for a subvariety V of Y :

Lemma 3.4. Suppose that V is a nonsingular subvariety of Y that intersects Z transversally in an irreducible variety $V \cap Z$, and that the restriction $A^*(V) \rightarrow A^*(V \cap Z)$ is also surjective. Let $\tilde{V} = Bl_Z V$. Then $A^*(\tilde{Y}) \rightarrow A^*(\tilde{V})$ is surjective, with kernel $J_{V/Y}$ if $V \cap Z$ is not empty, and kernel $(J_{V/Y}, E)$ if $V \cap Z$ is empty.

Proof. This is Lemma 5.4 in [FM]. \square

Lemma 3.5. Suppose that Z is the transversal intersection of nonsingular subvarieties V and W of Y , and that the restrictions $A^*(Y) \rightarrow A^*(V)$ and $A^*(Y) \rightarrow A^*(W)$ are also surjective, let $\tilde{V} = Bl_Z V$. Then

- (1) $A^*(\tilde{Y}) \rightarrow A^*(\tilde{V})$ is surjective, with kernel $(J_{V/Y}, P_{W/Y}(-E))$;
- (2) $A^*(\tilde{Y}) \rightarrow A^*(E \cap \tilde{V})$ is surjective, with kernel $(J_{Z/Y}, P_{W/Y}(-E))$.

Proof. This is Lemma 5.5 in [FM]. \square

Using the general results mentioned above we are able to express certain monomials that belong to the Chow ring $A^*(\tilde{Y})$ in terms of elements in $A^*(Y)$:

Lemma 3.6. Suppose that Z is the transversal intersection $D_1 \cap \dots \cap D_r$ of divisor classes D_1, \dots, D_r on Y and let $f \in A^*(Y)$ be an element of degree $d = \dim(Z)$. The following relation holds in $A^*(Bl_Z Y)$:

$$f.E^r = (-1)^{r-1} f.Z.$$

Proof. Multiply both sides of the equality

$$(D_1 - E) \dots (D_r - E) = 0$$

by f . For any element $g \in A^*(Y)$ of positive degree, the pull back $i_Z^*(fg)$ of fg along the inclusion $i_Z : Z \rightarrow Y$ is zero, which means that the product $fg.E$ is zero as well. This proves the claim. \square

We also state the non-vanishing criteria of the product $E_I.E_J$ for a pair of exceptional divisors E_I and E_J :

Proposition 3.7. Let $I, J \subset \{1, \dots, n\}$ be subsets satisfying $|I|, |J| \leq n-3$. The product $E_I.E_J \in A^2(\overline{U}_{n-1})$ is zero unless $I \subseteq J$ or $J \subseteq I$ or $I \cup J = \{1, \dots, n\}$.

Proof. If $I \cup J \neq \{1, \dots, n\}$, then $X_{I \cap J}$ is equal to the intersection $X_I \cap X_J$, and it is a proper subset of X_I and X_J both if $I \not\subseteq J$ and $J \not\subseteq I$. Under the assumption $I \not\subseteq J, J \not\subseteq I$ and $I \cup J \neq \{1, \dots, n\}$, the proper transforms of the subvarieties X_I and X_J become disjoint after blowing up along that of $X_{I \cap J}$. This means that the product $E_I.E_J \in A^2(\overline{U}_{n-1})$ is zero. \square

4. The tautological ring $R^*(C^n)$

Definition 4.1. Suppose $(C; O)$ is a fixed pointed elliptic curve, and let $n \in \mathbb{N}$ be a natural number. The tautological ring, $R^*(C^n) \subset A^*(C^n)$, is defined to be the \mathbb{Q} -subalgebra generated by the following classes:

$$a_i = \{(x_1, \dots, x_n) \in C^n : x_i = O\}, \quad d_{j,k} = \{(x_1, \dots, x_n) \in C^n : x_j = x_k\},$$

where $1 \leq i \leq n$ and $1 \leq j < k \leq n$. If we define $b_{j,k} := d_{j,k} - a_j - a_k$, then another set of generators for $R^*(C^n)$ is $\{a_i, b_{j,k} : 1 \leq i \leq n \text{ and } 1 \leq j < k \leq n\}$.

Proposition 4.2. (A) The space of relations in $R^*(C^n)$ is generated by the following ones:

$$a_i^2 = 0, \quad a_i b_{i,j} = 0, \quad b_{i,j} b_{i,k} = a_i b_{j,k}, \quad b_{i,j} b_{k,l} + b_{i,k} b_{j,l} + b_{i,l} b_{j,k} = 0,$$

where in each relation the indices are distinct.

(B) For any $0 \leq d \leq n$, the pairing $R^d(C^n) \times R^{n-d}(C^n) \rightarrow \mathbb{Q}$ is perfect.

Proof. We first verify that the relations above hold in $R^2(C^n)$. The relations $a_i^2 = a_i b_{i,j} = 0$ obviously hold. E. Getzler proved in [G] that the following relation holds in $A^2(\overline{M}_{1,4})$:

$$(1) \quad 12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_\beta = 0,$$

where the classes above are defined in [G].

In [P], R. Pandharipande gives a direct construction of Getzler's relation via a rational equivalence in the Chow group $A_2(\overline{M}_{1,4})$. If we restrict the relation (1) to the space $M_{1,4}^{ct}$, pull it back along the morphism $F : \overline{U}_3 \rightarrow M_{1,4}^{ct}$, and push it down to C^3 via the blow-down map, we get the relation

$$(2) \quad 12(a_1 b_{2,3} - b_{1,2} b_{1,3}) = 0.$$

Next, we deal with the last relation. Denote by $\pi : M_{1,5}^{ct} \rightarrow M_{1,4}^{ct}$ the morphism which forgets the fifth marking. If we pull back the restriction of the relation (1) by the map π , its pull-back along $F : \overline{U}_4 \rightarrow M_{1,5}^{ct}$ gives a relation whose push-down by the blow-down map to C^4 becomes

$$(3) \quad 12(b_{1,2} b_{3,4} + b_{1,3} b_{2,4} + b_{1,4} b_{2,3}) = 0.$$

For more details about the derivation of (2) and (3), please see the appendix.

Now, we study the pairing

$$R^d(C^n) \times R^{n-d}(C^n) \rightarrow \mathbb{Q}$$

for $0 \leq d \leq n$. From the relations above, we see that the tautological group $R^d(C^n)$ is generated by monomials of the form $v = a(v).b(v)$, where $a(v)$ is a product $\prod a_i$ of a_i 's for $i \in A_v$, and $b(v)$ is a product $\prod b_{j,k}$ of $b_{j,k}$'s, for $j, k \in B_v$, such that A_v and B_v are disjoint subsets of the set $\{1, \dots, n\}$ satisfying $d = |A_v| + \frac{1}{2}|B_v|$. Under this circumstance, the monomial v is said to be standard. To any standard monomial v we associate a dual element $v^* \in R^{n-d}(C^n)$, which is defined to be the product of all a_i 's, for $i \in \{1, \dots, n\} - A_v \cup B_v$, with $b(v)$. The following lemma enables us to study the pairing:

Lemma 4.3. *Let $v \in R^d(C^n)$ and $w \in R^{n-d}(C^n)$ be standard monomials. If the product $v.w$ is nonzero, then $B_v = B_w$, and the disjoint union of the sets A_v, A_w and $B_v = B_w$ is equal to the set $\{1, \dots, n\}$.*

Proof. By assumption, we obtain the following inequalities:

$$n = (|A_v| + \frac{1}{2}|B_v|) + (|A_w| + \frac{1}{2}|B_w|) \leq |A_v| + |A_w| + |B_v \cup B_w| \leq n.$$

This forces the inequalities to be equalities. The equality

$$(|B_v \cup B_w| - |B_v|) + (|B_v \cup B_w| - |B_w|) = 0$$

implies that $|B_v \cup B_w| = |B_v| = |B_w|$, which shows that $B_v = B_w$. The equality $|A_v| + |A_w| + |B_v| = n$ proves the second part of the claim. \square

So, after a suitable enumeration of generators for $R^d(C^n)$, the resulting intersection matrix of the pairing between standard monomials and their dual consists of square blocks along the main diagonal and the off-diagonal blocks are all zero. To prove that the pairing is perfect we need to study the square blocks on the main diagonal. These matrices and their eigenvalues are studied in [HW]. In particular, from their result it follows that the kernel of any such matrix is generated by relations of the form (3):

Lemma 4.4. *Let $m \geq 2$ be a natural number and S be the set of all standard monomials v of the form $b_{i_1, j_1} \dots b_{i_m, j_m}$ in $R^m(C^{2m})$. The kernel of the intersection matrix $(v.w)$ for v, w in S is generated by expressions of the form*

$$R_{\{i,j,k,l\}} := b_{i,j}b_{k,l} + b_{i,k}b_{j,l} + b_{i,l}b_{j,k},$$

where the indices are distinct elements varying over the set $\{1, \dots, 2m\}$.

Proof. The intersection matrix $(v.w)$ for $v, w \in S$ in [HW] is denoted by $T_r(x)$ for $r = m$ and $x = -2$. The S_{2m} -module generated by elements of S decomposes into the sum $\oplus_\lambda V_\lambda$, where λ varies over all partitions of $2m$ into even parts. For each such λ the space V_λ is an eigenspace of $T_r(x)$. The corresponding eigenvalue is zero when $\lambda \neq 2^m$ and it is $(-1)^m(m+1)!$ when $\lambda = 2^m$. We identify the space V_λ with a subspace of $R^m(C^{2m})$, defined below, which is generated by expressions of the form $R_{\{i,j,k,l\}}$, for $\lambda \neq 2^m$.

Recall that a tabloid is an equivalence class of numberings of a Young diagram, two being equivalent if corresponding rows contain the same entries. The tabloid determined by a numbering T is denoted $\{T\}$. The space V_λ is generated by elements of the form

$$v_T = \sum_{q \in C(T)} \text{sgn}(q) \{q.T\},$$

where $C(T)$ is the subgroup of S_{2m} consisting of permutations preserving the columns of T .

Note that the sum $R_{\{1, \dots, 2m\}} := \sum_{v \in S} v$ is a symmetric expression, which is clearly a linear combination of terms of the form R_T , where $|T| = 4$. This proves the claim when $\lambda = 2m$ gives the trivial representation. For other partitions λ we use the proven result for the symmetric relations. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_r > 0)$ be a partition of $2m$. For each numbering of a Young diagram T let T_i denote the subset of $\{1, \dots, 2m\}$ containing elements of the i^{th} row of T , for $i = 1, \dots, r$. Consider the product $P_T := \prod_{i=1}^r R_{T_i}$, where $R_{\{i,j\}}$ is defined to be $b_{i,j}$, while the other R_{T_i} 's are defined above when $|T_i| \geq 4$. Note that P_T doesn't change as T varies in an equivalence class $\{T\}$ since R_{T_i} 's are symmetric. This means that the assignment

$$v_T \rightarrow \sum_{q \in C(T)} \text{sgn}(q) P_{q,T},$$

is a well-defined S_{2m} -module morphism. This map is non-zero, hence an isomorphism onto its image. The result follows. \square

Since the relations of the form (3) hold in the tautological ring $R^*(C^n)$, we conclude that the pairing is perfect. This also shows that the relations stated in the proposition generate all relations in the tautological ring. \square

Remark 4.5. The tautological ring $R^*(C^n)$, for a smooth curve C of genus g , was defined by C. Faber and R. Pandharipande (unpublished) as the \mathbb{Q} -subalgebra of $A^*(C^n)$ generated by the standard classes K_i and $D_{i,j}$. They show that the image $RH^*(C^n)$ in cohomology is Gorenstein. In [GG] M. Green and P. Griffiths have shown that $R^*(C^2)$ is not Gorenstein, for C a generic complex curve of genus $g \geq 4$.

5. The tautological ring $R^*(\overline{U}_{n-1})$

Definition 5.1. Let Y be a blow-up of C^{n-1} at some step in the construction of \overline{U}_{n-1} , and denote by S_Y the collection of subsets of the set $\{1, \dots, n\}$ corresponding to the involved blow-up centers. The tautological ring $R^*(Y)$ of Y is defined to be the subalgebra of the Chow ring $A^*(Y)$ generated by the tautological classes in $R^*(C^{n-1})$ and the classes E_I , for I in S_Y . In particular, $R^*(\overline{U}_{n-1})$ is generated over $R^*(C^{n-1})$ by all E_I 's, where $|I| \leq n-3$.

5.1. Relations in $R^*(\overline{U}_{n-1})$.

- For subsets $I, J \subset \{1, \dots, n\}$, where $|I|, |J| \leq n-3$, the product $E_I \cdot E_J \in R^2(\overline{U}_{n-1})$ is zero unless

$$* \quad I \subseteq J, \quad \text{or } J \subseteq I, \quad \text{or } I \cup J = \{1, \dots, n\}.$$

- For any subset $I \subset \{1, \dots, n\}$, where $|I| \leq n-3$, consider the inclusion

$$i_I : X_I \rightarrow C^{n-1}.$$

The relations

$$\{x \cdot E_I = 0 : x \in \ker(i_I^* : R^*(C^{n-1}) \rightarrow R^*(X_I))\}$$

hold. Note that the kernel of i_I^* coincides with the kernel of the map

$$R^*(C^{n-1}) \rightarrow R^*(C^{n-1})$$

defined by $x \rightarrow x \cdot X_I$. This follows since $(i_I)_*(i_I)^*(x) = x \cdot X_I$ for $x \in R^*(C^{n-1})$, and $(i_I)_*$ is injective in our case.

- As we saw in the third section, in blowing-up the variety Y along a subvariety $Z \subset Y$, if the center Z can be written as the transversal intersection of the subvarieties V and W of Y , then the class $P_{W/Y}(-E_Z)$ is in the ideal $J_{\tilde{V}/\tilde{Y}}$. This means that the product $P_{W/Y}(-E_Z) \cdot E_{\tilde{V}}$ is zero, where $E_{\tilde{V}}$ is the class of the exceptional divisor of the blow-up along the subvariety \tilde{V} . We get a class of relations of this type by

writing the centers of blow-ups introduced in the construction of the space \overline{U}_{n-1} as transversal intersections in different ways. If the subvariety V can be written as a transversal intersection $V_1 \cap \dots \cap V_k$, we obtain the relation $P_{W/Y}(-E_Z) \cdot E_{V_1} \dots E_{V_k} = 0$.

- For each subvariety $Z \subset Y$ with a Chern polynomial $P_{Z/Y}(t)$, there is a relation

$$P_{Z/Y}(-E_Z) = 0,$$

where E_Z is the class of the exceptional divisor of the blow-up of Y along Z . These give another class of relations in $R^*(\overline{U}_{n-1})$. Note that for each subset I of $\{1, \dots, n\}$, a Chern polynomial $P_{X_I/C^{n-1}}$ of the subvariety X_I is in $R^*(C^{n-1})[t]$. This means that a Chern polynomial of its proper transform under later blow-ups belongs to $R^*(\overline{U}_{n-1})$. It follows from Lemma 3.3, which relates a Chern polynomial $P_{V/Y}(t)$ of a subvariety V to that of its proper transform \tilde{V} .

Example 5.2. Suppose $Y = C^5$.

- Let X_0 be the point $O^5 \in C^5$. Then $\ker(i^* : R^*(C^5) \rightarrow R^*(X_0))$ consists of all elements of positive degree. It follows that $a_i \cdot E_0 = b_{i,j} \cdot E_0 = 0$ for all i and j .
- Let $X_1 = a_2 a_3 a_4 a_5$. From $a_1 \cap X_1 = X_0$ we get the relation $(a_1 - E_0)E_1 = 0$. If $X_{1,2,3} = a_4 a_5$ and $X_{4,5,6} = d_{1,2} d_{1,3}$, then the relation $(a_1 - E_0)E_{1,2,3}E_{4,5,6} = 0$ follows from the equality $a_1 \cap X_{1,2,3} \cap X_{4,5,6} = X_0$.
- A Chern polynomial of the subvariety X_0 is $\prod_{i=1}^5 (a_i + t)$, from which we get the following relation:

$$\prod_{i=1}^5 (a_i - E_0) = a_1 a_2 a_3 a_4 a_5 - E_0^5 = 0.$$

There are few special cases of the relations above which will be useful in the definition of standard monomials and in defining the dual elements:

Lemma 5.3. *Let I be a subset of the set $\{1, \dots, n\}$ with at most $n - 3$ elements, containing n . For any $i \in I$ and $j, k \in \{1, \dots, n\} - I$, the following relations hold in $A^2(\overline{U}_{n-1})$:*

$$a_j \cdot E_I = a_k \cdot E_I, \quad b_{j,k} \cdot E_I = -2a_j \cdot E_I, \quad b_{i,k} \cdot E_I = \left(\sum_{J \subseteq I - \{i\}} E_J - a_i - a_j \right) \cdot E_I.$$

Proof. Recall that E_I is the exceptional divisor of the blow-up along the proper transform of the subvariety

$$X_I = \bigcap_{j \neq r \in \{1, \dots, n\} - I} d_{j,r} = \bigcap_{k \neq r \in \{1, \dots, n\} - I} d_{k,r}.$$

The equality $a_j \cdot E_I = a_k \cdot E_I$ follows since

$$a_j - a_k \in \ker(i_I^* : R^*(C^{n-1}) \rightarrow R^*(X_I)),$$

where $i_I : X_I \rightarrow C^{n-1}$ denotes the inclusion.

We give another proof as well: from the following equality

$$X_I \cap a_j = X_I \cap a_k = X_{I - \{n\}},$$

we obtain the relation

$$(a_j - \sum_{J \subseteq I - \{n\}} E_J) E_I = (a_k - \sum_{J \subseteq I - \{n\}} E_J) E_I = 0.$$

This gives the first relation after canceling out $(\sum_{J \subseteq I - \{n\}} E_J) E_I$ on both sides.

The second equality results from the definition $b_{j,k} = d_{j,k} - a_j - a_k$, the relation $d_{j,k} \cdot E_I = 0$, and from the previous case.

To prove the last statement, first note that $b_{i,k} = d_{i,k} - a_i - a_k$, by definition. From the equality $X_I \cap d_{i,k} = X_{I-\{i\}}$, we get the relation

$$(d_{i,k} - \sum_{J \subseteq I - \{i\}} E_J).E_I = 0.$$

We conclude that

$$b_{i,k}.E_I = (\sum_{J \subseteq I - \{i\}} E_J - a_i - a_k).E_I,$$

which proves the last statement, using that $a_j.E_I = a_k.E_I$. \square

5.2. Standard monomials. The existence of the relations stated above makes it possible to obtain a smaller set of generators for the tautological ring $R^*(\overline{U}_{n-1})$. Any monomial $v \in R^d(\overline{U}_{n-1})$ can be written as a product $a(v)b(v)E(v)$, where $a(v)$ is a product of a_i 's, $b(v)$ is a product of $b_{j,k}$'s, and $E(v)$ is a product of exceptional divisors. To simplify the enumeration of the generators for $R^*(\overline{U}_{n-1})$, we introduce the directed graph associated to a monomial:

Definition 5.4. Let $v = a(v)b(v)E_{I_1}^{i_1} \dots E_{I_m}^{i_m}$, where $i_r \neq 0$ for $r = 1, \dots, m$ and $I_1 < \dots < I_m$, be a monomial. The directed graph $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$ associated to v is defined by the following data:

- A set $V_{\mathcal{G}}$ and a one-to-one correspondence between members of $V_{\mathcal{G}}$ and members of the set $\{1, \dots, m\}$. Elements of $V_{\mathcal{G}}$ are called the vertices of \mathcal{G} .
- A set $E_{\mathcal{G}} \subset V_{\mathcal{G}} \times V_{\mathcal{G}}$ consisting of all pairs (r, s) , where I_s is a minimal element of the set

$$\{I_i : I_r \subset I_i\}$$

with respect to inclusion. Elements of $E_{\mathcal{G}}$ are called the edges of \mathcal{G} .

For a vertex $i \in V_{\mathcal{G}}$, the closure $\bar{i} \subset V_{\mathcal{G}}$ is defined to be the subset

$$\{r \in V_{\mathcal{G}} : I_i \subseteq I_r\}$$

of $V_{\mathcal{G}}$. The degree $\deg(i)$ of i is defined to be the number of the elements of the set

$$\{j \in V_{\mathcal{G}} : (i, j) \in E_{\mathcal{G}}\}.$$

A vertex $i \in V_{\mathcal{G}}$ is called a *root* of \mathcal{G} if I_i is minimal with respect to inclusion of sets. Maximal vertices of \mathcal{G} are called *external* and all the other vertices will be called *internal*.

In the following, we use the letters I_1, \dots, I_m to denote the vertices of \mathcal{G} .

Remark 5.5. We can define a directed graph associated to any collection of subsets of the set $\{1, \dots, n\}$ in a similar way. In general there may be a loop in the resulting graph after forgetting the directions. Loops don't occur when the collection consists of only proper subsets I of the set $\{1, \dots, n\}$ such that for any two distinct members I, J of the collection one of the conditions in $*$ holds. Hence, we refer to \mathcal{G} as the associated forest of the monomial v , or of the collection $\{I_1, \dots, I_m\}$.

The next lemma turns out to be useful in defining the dual element:

Lemma 5.6. Suppose that I_1, \dots, I_m are proper subsets of the set $\{1, \dots, n\}$, containing at most $n - 3$ elements, with the property that each pair I_r and I_s satisfy $*$. Let \mathcal{G} be the associated forest. If $n \notin \cap_{r=1}^m I_r$, then there is a unique root of \mathcal{G} not containing n .

Proof. By assumption, there is a root I_r such that $n \notin I_r$. Uniqueness follows since for any two roots I_r, I_s of \mathcal{G} the equality $I_r \cup I_s = \{1, \dots, n\}$ holds. This means that their complements I_r^c, I_s^c are disjoint. Hence, n belongs to the complement of at most one root. \square

Definition 5.7. Let v be as in Definition 5.4, \mathcal{G} be the associated forrest, and J_1, \dots, J_s , for some $s \leq m$, be roots of \mathcal{G} . For each $1 \leq r \leq s$ such that $n \in J_r$, let $j_r \in \{1, \dots, n\} - J_r$ be the smallest element. The subset S of the set $\{1, \dots, n-1\}$ is defined as follows:

- If $n \in \cap_{r=1}^m I_r$, put

$$S := \{j_1, \dots, j_s\} \cup (\cap_{r=1}^m I_r - \{n\}),$$

- if $n \notin \cap_{r=1}^m I_r$, let J_1 be the unique root of \mathcal{G} not containing n . In this case

$$S := \{j_2, \dots, j_s\} \cup (\cap_{r=1}^m I_r).$$

The monomial v is said to be standard if

- The monomial $a(v)b(v) \in R^*(C^S)$ is in standard form according to the definition given in the forth section.
- For each r we have that

$$i_r \leq \min(n-2 - |I_r|, |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 2).$$

To prove that the standard monomials generate the tautological ring, we define an ordering on the *polynomial ring*

$$R := \mathbb{Q}[a_i, b_{j,k}, E_I : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1, I \subset \{1, \dots, n\}, \text{ where } |I| \leq n-3].$$

Definition 5.8. Let $I, J \subset \{1, \dots, n\}$, we say that $I < J$ if

- $|I| < |J|$
- or, if $|I| = |J|$ and the smallest element in $I - I \cap J$ is less than the smallest element of $J - I \cap J$.

Put an arbitrary total order on monomials in

$$\mathbb{Q}[a_i, b_{j,k} : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1].$$

Suppose $v_1, v_2 \in R$ are monomials. We say that $v_1 < v_2$ if we can write them as

$$v_1 = a(v_1)b(v_1) \prod_{r=1}^{r_0} E_{I_r}^{i_r} \cdot E, \quad \text{and } v_2 = a(v_2)b(v_2) \prod_{r=1}^{r_0} E_{I_r}^{j_r} \cdot E,$$

where $E = \prod_{r=r_0+1}^m E_{I_r}^{i_r}$, for $I_1 < \dots < I_m$, and $i_{r_0} < j_{r_0}$;

or, if $r_0 = 0$ and $a(v_1)b(v_1) < a(v_2)b(v_2)$.

Furthermore, we say that $v_1 \ll v_2$, if for any factor E_I of v_2 we have that $v_1 < E_I \cdot v_2$. Note that $v_1 \ll v_2$ implies that $v_1 < v_2$.

Proposition 5.9. The tautological ring $R^*(\overline{U}_{n-1})$ of \overline{U}_{n-1} is generated by standard monomials.

Proof. Let v be a monomial given as in Definition 5.4. We have seen that any monomial in $R^d(C^{n-1})$ can be written in standard form for $0 \leq d \leq n-1$. By Lemma 5.3, we may assume that $a(v)b(v)$ is an element of the tautological ring $R^*(C^S)$ of C^S , where S is defined according to the Definition 5.7. The statement is proven using induction and from the following observations:

- From the last class of relations in 5.1, for any subset I_r of the set $\{1, \dots, n\}$, where $|I_r| \leq n-3$, we can write $E_{I_r}^{n-1-|I_r|}$ as a sum of elements which are strictly less than it.
- Let $\{J_1, \dots, J_s\}$ be the set of minimal elements of the set

$$\{I_i : I_r \subset I_i, \text{ where } 1 \leq i \leq m\}$$

From the third class of relations in 5.1, the monomial $E_{I_r}^j \prod_{i=1}^s E_{J_i}$ can be written as a sum of terms which are strictly less than it, where $j = |\cap_{i=1}^s J_i| - |I_r| + s - 1$.

□

5.3. Definition of the dual element. Now suppose $v \in R^d(\overline{U}_{n-1})$ is an element of degree d written in standard form. Below, we define the element v^* , which is an element of $R^{n-1-d}(\overline{U}_{n-1})$. As we will see, the property $v^{**} = v$ holds. This shows that there is a one-to-one correspondence between standard monomials in degree d and $n - 1 - d$.

Definition 5.10. Suppose $v = a(v)b(v)E(v)$ is a standard monomial, where $a(v)b(v)$ is in the tautological ring $R^*(C^{n-1})$ of C^{n-1} , and

$$E(v) = \prod_{r=1}^m E_{I_r}^{i_r},$$

where $i_r \neq 0$ for $r = 1, \dots, m$, and $I_1 < \dots < I_m$. Let \mathcal{G} be the associated forest, and J_1, \dots, J_s , for some $s \leq m$, be the roots of \mathcal{G} . For each $1 \leq r \leq s$ such that $n \in J_r$, let $j_r \in \{1, \dots, n\} - J_r$ be the smallest element. The subset T of the set $\{1, \dots, n - 1\}$ is defined as follows:

- If $n \in \cap_{r=1}^m I_r$ put

$$T := \{j_1, \dots, j_s\} \cup (\cap_{r=1}^m I_r) - (A_v \cup B_v \cup \{n\}),$$

- if $n \notin \cap_{r=1}^m I_r$, let J_1 be the unique root of \mathcal{G} not containing n . In this case

$$T := \{j_2, \dots, j_s\} \cup (\cap_{r=1}^m I_r) - (A_v \cup B_v).$$

For each $1 \leq r \leq m$, define j_r to be

$$\begin{cases} |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 1 - i_r & I_r \text{ is an internal vertex of } \mathcal{G} \\ n - 1 - |I_r| - i_r & I_r \text{ is an external vertex of } \mathcal{G}. \end{cases}$$

We define $v^* = a(v^*)b(v^*)E(v^*)$, where

$$a(v^*) = \prod_{i \in T} a_i, \quad b(v^*) = b(v), \quad E(v^*) = \prod_{r=1}^m E_{I_r}^{j_r}.$$

Remark 5.11. We verify that the dual element v^* is well-defined. We need to show that the integers j_r are non-negative. These integers are indeed positive numbers for $r = 1, \dots, m$. In the definition of standard monomials we have seen that

$$i_r \leq |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 2.$$

This shows that $j_r \geq 1$ when $i_r > 0$ and I_r is an internal vertex. The case of external vertices is treated using the inequality $i_r \leq n - 2 - |I_r|$.

The following corollary follows from Definition 5.10.

Corollary 5.12. Suppose $v \in R^d(\overline{U}_{n-1})$ is a standard monomial and let $v^* \in R^{n-1-d}(\overline{U}_{n-1})$ be its dual. Then v^* is a standard monomial, and furthermore $v^{**} = v$.

The next lemma will be useful in the proof of the Proposition 5.16 and the identity 5:

Lemma 5.13. Let $v = a(v)b(v)E(v)$ be as in the Definition 5.10, and \mathcal{G} be the associated forest. For a vertex $i \in V_{\mathcal{G}}$ corresponding to the subset I_i of the set $\{1, \dots, n\}$, the equality

$$\sum_{\bar{i}} (i_r + j_r) = n - 1 - |I_i|$$

holds. Here \bar{i} is the closure of i in \mathcal{G} defined in the Definition 5.4.

Proof. It is immediate from the definition of the j_r 's above. □

5.4. The pairing $R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1})$. In the previous part, we defined dual elements for standard monomials. Below, we will see that the resulting intersection matrix between the standard monomials and their duals consists of square blocks on the main diagonal, whose entries are, up to a sign, intersection numbers in $R^{|S|}(C^S)$, for certain sets S , and all blocks under the diagonal are zero. To prove the stated properties of the intersection matrix, we introduce a natural filtration¹ on the tautological ring.

Definition 5.14. Let v be a standard monomial as given in Definition 5.7, and let J_1, \dots, J_s be roots of the associated forest. Define $p(v)$ to be the degree of the element

$$a(v)b(v) \cap_{r=1}^s X_{J_r} \in A^*(C^{n-1}),$$

which is the same as the integer

$$\deg a(v)b(v) + n - |\cap_{r=1}^s J_r| - s.$$

The subspace $F^p R^*(\overline{U}_{n-1})$ of the tautological ring is defined to be the \mathbb{Q} -vector space generated by standard monomials v satisfying $p(v) \geq p$.

Proposition 5.15. (a) For any integer p , we have that $F^{p+1} R^*(\overline{U}_{n-1}) \subseteq F^p R^*(\overline{U}_{n-1})$.

(b) Let $v \in F^p R^*(\overline{U}_{n-1})$ and $w \in R^d(\overline{U}_{n-1})$ be such that $w \ll v$. If $p + d \geq n$, then $v.w$ is zero. In particular, $F^n R^*(\overline{U}_{n-1})$ is zero.

Proof. The first statement is immediate from Definition 5.14. Let us prove (b). Let v be given as in Definition 5.7. Denote by Y the blow-up of C^{n-1} corresponding to the collection

$$\{J \subset \{1, \dots, n\} : J < J_r \text{ for } 1 \leq r \leq s\}$$

Note that the dimension $\dim \cap_{r=1}^s X_{J_r}$ of the transversal intersection $\cap_{r=1}^s X_{J_r}$ is equal to $|\cap_{r=1}^s J_r| + s - 1$. The product

$$a(v)b(v) \cdot \prod_{r=1}^s \tilde{X}_{J_r} \cdot w \in R^*(Y)$$

is zero since its degree, which is

$$\deg(a(v)b(v)) + n - |\cap_{r=1}^s J_r| - s + d,$$

is at least n , by assumption. This means that the product

$$a(v)b(v) \cdot \prod_{r=1}^s E_{J_r} \cdot w \in R^*(\overline{U}_{n-1}),$$

which is a factor of $v.w$, is zero as well. \square

Using the proven lemma we are able to prove the following vanishing result:

Proposition 5.16. Suppose $v_1, v_2 \in R^d(\overline{U}_{n-1})$ are standard monomials satisfying $E(v_1) < E(v_2)$. Then $v_1.v_2^* = 0$.

Proof. It is enough to write $v_1.v_2^*$ as a product $v.w$, for $v, w \in R^*(\overline{U}_{n-1})$ satisfying the properties given in the proposition 5.15. To find v and w , let v_1, v_2 be given as in Definition 5.8, and denote by $\{J_1, \dots, J_s\}$ the set of roots of the graph associated to the monomial $E = \prod_{r=r_0+1}^m E_{I_r}^{i_r}$. By relabeling the roots we may assume that there is an $s_0 \geq 0$ such that $I_{r_0} \subset J_r$ for $1 \leq r \leq s_0$, and the equality $I_{r_0} \cup J_r = \{1, \dots, n\}$ holds for $s_0 < r \leq s$. Let w be the product of all monomials in $v_1.v_2^*$ which are strictly less than $E_{I_{r_0}}$ and v be the product

¹The definition of this filtration on the tautological ring was formulated after a question of E. Looijenga.

of the other factors, so that $v_1.v_2^* = v.w$. Notice that $w \ll v$, by the definition of v and w . The degree d of w is computed using Lemma 5.13:

$$\begin{aligned} d &= n + j_{r_0} - i_{r_0} + s - s_0 - |I_{r_0}^c| - \sum_{r=s_0+1}^s |J_r^c| = j_{r_0} - i_{r_0} + s - s_0 + |I_{r_0} \cap J_{s_0+1} \cap \dots \cap J_s| \\ &\geq s - s_0 + 1 + |I_{r_0} \cap J_{s_0+1} \cap \dots \cap J_s| = n - p(v). \end{aligned}$$

From $d + p(v) \geq n$ we see that the product $v.w$ is zero. \square

To study the blocks on the main diagonal we proceed as follows: we first prove an identity which reduces the number of exceptional divisors in the product for certain monomials.

Let Y be a blow-up of C^{n-1} at some step in the construction of \overline{U}_{n-1} . Suppose that

$$V_1 \cap \dots \cap V_k \cap W = Z$$

is a transversal intersection of tautological classes, where $W = D_1 \cap \dots \cap D_r$ is a transversal intersection of divisors $D_1, \dots, D_r \in R^1(Y)$, and let $f \in R^*(Y)$ be an element of degree $d = \dim(Z)$. Denote by E_Z the exceptional divisor of the blow-up $Bl_Z Y$ of Y along Z and by E_{V_1}, \dots, E_{V_k} those of the blow-up \tilde{Y} of $Bl_Z Y$ along the proper transform of the subvarieties V_1, \dots, V_k .

It follows from the Lemma 3.5 that $P_{W/Y}(-E_Z) \in J_{\tilde{Y}/\tilde{Y}}$, for $V = V_1 \cap \dots \cap V_k$. Using the same argument as in Lemma 3.6 we observe that the equality

$$f.E_Z^r E_{V_1} \dots E_{V_k} = (-1)^{r-1} f.W.E_{V_1} \dots E_{V_k}$$

holds in $R^{r+d+k}(\tilde{Y})$.

If the codimension of the subvariety V_i is r_i and that of Z is r_0 , then from the proven result in Lemma 3.6 one gets the following identity:

$$(4) \quad f.E_Z^r E_{V_1}^{r_1} \dots E_{V_k}^{r_k} = (-1)^{r_0-k-1} f.Z.$$

Notice that this identity reduces the computation of certain monomials containing the exceptional divisors to one which belongs to the Chow ring $A^*(Y)$. We now use this identity to compute the numbers occurring on the main diagonal of the intersection matrix.

Let I_1, \dots, I_m be subsets of the set $\{1, \dots, n\}$ containing at most $n-3$ elements such that for every pair I_r and I_s the property $*$ holds. Let \mathcal{G} be the associated forest with roots J_1, \dots, J_s , and for each $1 \leq r \leq s$ such that $n \in J_r$, let $j_r \in \{1, \dots, n\} - J_r$ be the smallest element. If $n \notin \cap_{r=1}^m I_r$, let J_1 be the unique root of \mathcal{G} not containing n . Define

$$E := \prod_{r=1}^m E_{I_r}^{i_r},$$

where

$$i_r = \begin{cases} |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 1 & I_r \text{ is an internal vertex of } \mathcal{G} \\ n - 1 - |I_r| & I_r \text{ is an external vertex of } \mathcal{G}. \end{cases}$$

Consider an element $f \in \mathbb{Q}[a_i, b_{i,j} : i, j \in S]$, of degree $|\cap_{r=1}^m I_r| + s - 1$, where

$$S = \begin{cases} \{j_1, \dots, j_s\} \cup (\cap_{r=1}^m I_r - \{n\}) & n \in \cap_{r=1}^m I_r \\ \{j_2, \dots, j_s\} \cup (\cap_{r=1}^m I_r) & n \notin \cap_{r=1}^m I_r. \end{cases}$$

Then from the identity (4) it follows that

$$(5) \quad f.E = (-1)^\varepsilon . f. \prod_{i \in \{1, \dots, n-1\} - S} a_i.$$

where $\varepsilon = n + |\cap_{r=1}^m I_r| + \sum_{i \in V(G)} \deg(i)$, using Lemma 5.13.

Note that for any $v \in R^d(\overline{U}_{n-1})$, the product $E := E(v)E(v^*)$ is in the form given above according to the Definition 5.10.

Theorem 5.17. *For any $0 \leq d \leq n-1$, the pairing*

$$R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1}) \rightarrow \mathbb{Q}$$

is perfect.

Proof. Let $A := \{v_1, \dots, v_r\} \subset R^d(\overline{U}_{n-1})$, where $v_1 < \dots < v_r$, be the set of standard monomials of degree d , and $\{v_1^*, \dots, v_r^*\} \subset R^{n-1-d}(\overline{U}_{n-1})$ be the set of their duals defined above. For a monomial

$$E \in \mathbb{Q}[E_I : I \subset \{1, \dots, n\} \text{ and } |I| \leq n-3],$$

define

$$A_E := \{v \in A : E(v) = E\}.$$

Let \mathcal{G} be the graph associated to the monomial E , and define S as in Definition 5.7. For $v_i, v_j \in A_E$ the number

$$v_i.v_j^* \in R^{n-1}(\overline{U}_{n-1}) = \mathbb{Q}$$

is the same as

$$(-1)^\varepsilon a(v_i)b(v_i).a(v_j^*)b(v_j^*) \in R^S(C^S) = \mathbb{Q},$$

by the identity (5), where $\varepsilon = n + |\cap_{r=1}^m I_r| + \sum_{i \in V(\mathcal{G})} \deg(i)$.

This means that the intersection matrices $(v_i.v_j^*)$ and $(a(v_i)b(v_i).a(v_j^*)b(v_j^*))$, for v_i, v_j in the set A_E , are the same up to a sign after the identifications above. From the study of the tautological ring $R^*(C^S)$ of C^S , we know that the kernel of the matrix above is generated by relations in $R^*(C^S)$. After choosing a basis for $R^{d-\deg(E)}(C^S)$, the resulting matrix is invertible. It means that the intersection matrix of the pairing

$$R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1}) \rightarrow \mathbb{Q}$$

with this choice of basis elements for the tautological groups consists of invertible blocks on the main diagonal and zero blocks under the diagonal, hence is invertible. This proves the claim. \square

6. The tautological ring $R^*(M_{1,n}^{ct})$

In the first part we obtained the morphism $F : \overline{U}_{n-1} \rightarrow M_{1,n}^{ct}$, induced from the family of curves $\pi : \overline{U}_n \rightarrow \overline{U}_{n-1}$. The morphism F induces a ring homomorphism

$$F^* : A^*(M_{1,n}^{ct}) \rightarrow A^*(\overline{U}_{n-1}).$$

For a subset $J \subset \{1, \dots, n\}$, the pull back $F^*(D_J)$ of the divisor class D_J is a subvariety of \overline{U}_{n-1} for which the fiber $\pi^{-1}(P)$ is a nodal curve of type given by D_J , when P is its generic point. It follows that P is a point of the proper transform of the subvariety X_I , where $I := \{1, \dots, n\} - J$, when $|J| \geq 3$, and belongs to the proper transform of the divisors $a_i, d_{j,k}$ if $J = \{i, n\}, \{j, k\}$, for $1 \leq i \leq n-1$ and $1 \leq j < k \leq n-1$. But the proper transform of X_I is equal to E_I , when $|I| \leq n-3$, and those of the divisors $a_i, d_{j,k}$ are $a_i - \sum_{i \notin I \subset \{1, \dots, n-1\}} E_I$ and $d_{j,k} - \sum_{I \subset \{1, \dots, n\} - \{j, k\}} E_I$ respectively, for $1 \leq i \leq n-1$ and $1 \leq j < k \leq n-1$. This means that

$$F^*(D_{\{i, n\}}) = a_i - \sum_{i \notin I \subset \{1, \dots, n-1\}} E_I \text{ for } 1 \leq i \leq n-1,$$

$$F^*(D_{\{j, k\}}) = d_{j,k} - \sum_{I \subset \{1, \dots, n\} - \{j, k\}} E_I \text{ for } 1 \leq j < k \leq n-1,$$

$$F^*(D_J) = E_{\{1, \dots, n\} - J} \text{ for } |J| \geq 3.$$

From this we see that the pull-back homomorphism F^* sends tautological classes to tautological classes, and defines a ring homomorphism

$$F^* : R^*(M_{1,n}^{ct}) \rightarrow R^*(\overline{U}_{n-1}).$$

If we rewrite the expressions above, we get that

$$a_i = F^*\left(\sum_{i,n \in I} D_I\right) \text{ for } 1 \leq i \leq n-1, \quad d_{i,j} = F^*\left(\sum_{j,k \in I} D_I\right) \text{ for } 1 \leq j < k \leq n-1,$$

$$E_I = F^*(D_{\{1, \dots, n\} - I}) \text{ for } |I| \leq n-3.$$

This shows that F^* is a surjection. We prove that F^* is injective by extending the function

$G : \{a_i, d_{j,k}, E_I : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1, I \subset \{1, \dots, n\} \text{ and } |I| \leq n-3\} \rightarrow R^*(M_{1,n}^{ct})$,
defined by the rule

$$G(a_i) = \sum_{i,n \in I} D_I \text{ for } 1 \leq i \leq n-1, \quad G(d_{j,k}) = \sum_{j,k \in I} D_I \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq j < k \leq n-1,$$

$$G(E_I) = D_{I^c} \text{ for } |I| \leq n-3$$

to a ring homomorphism

$$G : R^*(\overline{U}_{n-1}) \rightarrow R^*(M_{1,n}^{ct}).$$

This is done by verifying that all relations between elements $a_i, d_{j,k}, E_I$'s on the left hand side hold between classes on the right hand side. To simplify the notation, we drop the letter G for tautological classes in $R^*(M_{1,n}^{ct})$. For instance, we write $a_i = \sum_{i,n \in I} D_I$ and $E_I = D_{I^c}$ for a subset $I \subset \{1, \dots, n\}$ with $|I| \leq n-3$.

Let us introduce the following notation: suppose S is a subset of the set $\{1, \dots, n\}$. By $M_{1,S}^{ct}$, we mean the moduli space of stable curves of genus one of compact type whose marking set is S . Let

$$\pi_S : M_{1,n}^{ct} \rightarrow M_{1,S}^{ct}$$

be the projection which forgets the markings in $\{1, \dots, n\} - S$ and contracts unstable components.

- We first deal with relations among generators of $R^*(C^{n-1})$: notice that

$$a_i = \pi_{\{i,n\}}^*(D_{\{i,n\}}), \quad d_{j,k} = \pi_{\{j,k\}}^*(D_{\{j,k\}}), \quad b_{j,k} = \pi_{\{j,k,n\}}^*(D_{\{j,k\}} - D_{\{j,n\}} - D_{\{k,n\}} - D_{\{j,k,n\}}).$$

From the relations $D_{\{i,n\}}^2 = D_{\{j,k\}}^2 = 0$ in $R^2(M_{1,\{i,n\}}^{ct})$ and $R^2(M_{1,\{j,k\}}^{ct})$, we obtain that the relations $a_i^2 = d_{j,k}^2 = 0$ hold in $R^2(M_{1,n}^{ct})$, for $1 \leq i \leq n-1$ and $1 \leq j < k \leq n-1$. On the other hand, the relation

$$(D_{\{i,j\}} - D_{\{i,n\}} - D_{\{j,n\}} - D_{\{i,j,n\}})(D_{\{i,n\}} + D_{\{i,j,n\}}) = 0 \in R^2(M_{1,\{i,j,n\}}^{ct})$$

says that $a_i b_{i,j} = 0$. From the relation $d_{j,k}^2 = 0$ obtained above, we get that $b_{j,k}^2 = -2a_j a_k$. Now suppose that i, j, k, l are distinct elements of the set $\{1, \dots, n-1\}$. As we saw in the forth section, the relation (1) in $R^2(M_{1,\{i,j,k,n\}}^{ct})$ can be written as

$$12(a_i b_{j,k} - b_{i,j} b_{i,k}) = 0.$$

The relation

$$12(b_{i,j} b_{k,l} + b_{i,k} b_{j,l} + b_{i,l} b_{j,k}) = 0$$

is the pull-back of the relation above to $R^2(M_{1,\{i,j,k,l,n\}}^{ct})$ along the morphism

$$\pi_{\{i,j,k,n\}} : M_{1,\{i,j,k,l,n\}}^{ct} \rightarrow M_{1,\{i,j,k,n\}}^{ct}.$$

This shows that the classes $a_i, b_{j,k} \in R^*(M_{1,n}^{ct})$ satisfy all relations among $a_i, b_{j,k} \in R^*(\overline{U}_{n-1})$.

- Note that the following

$$D_I.D_J \neq 0 \Rightarrow I \subseteq J \text{ or } J \subseteq I \text{ or } I \cap J = \emptyset$$

is true. But this can be written as

$$E_I.E_J \neq 0 \Rightarrow I \subseteq J \text{ or } J \subseteq I \text{ or } I \cup J = \{1, \dots, n\}.$$

This proves that the E_I 's in $R^*(M_{1,n}^{ct})$ satisfy the first class of relations between E_I 's in $R^*(\overline{U}_{n-1})$ obtained above.

- For any $I \subset \{1, \dots, n\}$ with $|I| \leq n-3$, we found the relations $x.E_I = 0$ for $x \in \ker(i_I^*)$, where $i_I : X_I \rightarrow C^{n-1}$ denotes the inclusion. If $n \notin I$, then $\ker(i_I^*)$ is generated by divisor classes $a_i, b_{i,j}$, where $i \in J := \{1, \dots, n-1\} - I$, and $j \in \{1, \dots, n-1\}$ is different from i . Let us see that $a_i.E_I = 0$ in this case:

$$a_i.E_I = D_J^2 + \sum_{J_0: i, n \in J_0 \subset J} D_{J_0}.D_J + \sum_{J_0: i, n \in J_0, J \subset J_0} D_{J_0}.D_J.$$

But this expression is zero from the following known formula for ψ classes in genus zero and one:

Proposition 6.1. (a) The following relation holds in $A^1(\overline{M}_{0,n})$:

$$\psi_i = \sum_{j, k \notin I, i \in I, |I| \geq 2} D_I$$

for some fixed distinct $j, k \in \{1, \dots, n\} - \{i\}$.

(b) The following relation holds in $A^1(M_{1,n}^{ct})$:

$$\psi_i = \sum_{i \in I, |I| \geq 2} D_I.$$

Proof. (a) is Proposition 1.6 in [AC]. To prove (b), it is enough to restrict the divisor classes given in Proposition 1.9 of [AC], to the space $M_{1,n}^{ct}$. \square

If $i \in \{1, \dots, n-1\} - I$, and $j \in \{1, \dots, n-1\}$ is distinct from i , we saw that $a_i.E_I = 0$, and by the same argument as above we see that $a_j.E_I = d_{i,j}.E_J$, from which it follows that

$$b_{i,j}.E_I = (d_{i,j} - a_i - a_j).E_I = 0.$$

The case $n \in I$ is proven by the same argument.

- We get a relation $P_{W/Y}(-E_Z).E_{V_1} \dots E_{V_k} = 0$, when the subvariety Z is a transversal intersection $V_1 \cap \dots \cap V_k \cap W$. After possibly relabeling the indices, we can assume that

$$Z = X_{I_0}, \quad V_i = X_{I_i}, \text{ for } 1 \leq i \leq k, \quad W = \prod_{i=r_0+1}^{r_1} a_i \cdot \prod_{j=1}^{k-1} a_{r_j+1},$$

where $r_0 \leq r_1 < \dots < r_k < n$, and $I_0 = \{1, \dots, r_0\}$, $I_i^c = \{r_i + 1, \dots, r_{i+1}\}$, for $1 \leq i < k$, and $I_k^c = \{r_k + 1, \dots, n\}$. Let us prove that

$$P_{W/C^{n-1}}(-\sum_{J \subseteq I_0} E_I)E_{I_1} \dots E_{I_k} = 0 \in R^{r_1-r_0+2k-1}(M_{1,n}^{ct})$$

by showing that any monomial in the expansion of this expression is zero. Let

$$\prod_{i=r_0+1}^{r_1} E_{J_i} \cdot \prod_{j=1}^{k-1} E_{J_{r_j+1}}.E_{I_1} \dots E_{I_k},$$

where

$$i, n \in J_i^c \text{ for } r_0 + 1 \leq i \leq r_1, \text{ and } r_j + 1, n \in J_j^c \text{ for } 1 \leq j \leq k - 1,$$

be any such monomial.

For $r_0 + 1 \leq i \leq r_1$, if the product $E_{J_i} \cdot E_{I_k}$ is non-zero, then $J_i \subseteq I_k - \{i\}$. For $1 \leq j \leq k - 1$, if the product $E_{J_{r_j+1}} \cdot E_{I_j}$ is non-zero, then $J_{r_j+1} \subseteq I_j$. On the other hand, since $n \notin J_i, J_{r_j+1}$ for all i, j , the product $E_{J_{i_1}} \cdot E_{J_{i_2}}$ is non-zero only if $J_{i_1} \subseteq J_{i_2}$ or $J_{i_2} \subseteq J_{i_1}$. It follows that the subsets J_i 's are totally ordered by inclusion, which means that their intersection is one of them. We conclude that for some i the inclusion $J_i \subseteq I_0$ holds. But this term is excluded from expression above, whence the product is zero.

- For any subset $I \subset \{1, \dots, n\}$, where $|I| \leq n-3$, we prove that $P_{X_I/C^{n-1}}(-\sum_{J \subseteq I} E_J)$ is zero by the same argument as in the previous case, by showing that the monomials occurring in the expansion of the expression above are all zero.

The argument above shows that F^* is an isomorphism, and hence, we have the following result:

Theorem 6.2. *The ring homomorphism $F^* : R^*(M_{1,n}^{ct}) \rightarrow R^*(\overline{U}_{n-1})$ is an isomorphism. In particular, for any $0 \leq d \leq n-1$, the pairing*

$$R^d(M_{1,n}^{ct}) \times R^{n-1-d}(M_{1,n}^{ct}) \rightarrow \mathbb{Q}$$

is perfect. In other words, $R^(M_{1,n}^{ct})$ is a Gorenstein ring.*

7. APPENDIX: DERIVATION OF THE RELATIONS (2) AND (3)

In this appendix we explain why the relations (2) and (3) follow from Getzler's relation (1). First note that the restriction of the relation (1) to the space $M_{1,4}^{ct}$ becomes

$$(6) \quad 12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} = 0.$$

Then we compute the pull-back of the classes above to the space \overline{U}_3 along the morphism

$$F : \overline{U}_3 \rightarrow M_{1,4}^{ct}.$$

Recall that

$$\begin{aligned} \delta_{2,2} &= D_{\{1,2\}}D_{\{3,4\}} + D_{\{1,3\}}D_{\{2,4\}} + D_{\{1,4\}}D_{\{2,3\}}, \\ \delta_{2,3} &= D_{\{1,2\}}D_{\{1,2,3\}} + D_{\{1,2\}}D_{\{1,2,4\}} + D_{\{1,3\}}D_{\{1,2,3\}} + D_{\{1,3\}}D_{\{1,3,4\}} \\ &\quad + D_{\{1,4\}}D_{\{1,2,4\}} + D_{\{1,4\}}D_{\{1,3,4\}} + D_{\{2,3\}}D_{\{1,2,3\}} + D_{\{2,3\}}D_{\{2,3,4\}} \\ &\quad + D_{\{2,4\}}D_{\{1,2,4\}} + D_{\{2,4\}}D_{\{2,3,4\}} + D_{\{3,4\}}D_{\{1,3,4\}} + D_{\{3,4\}}D_{\{2,3,4\}}, \\ \delta_{2,4} &= D_{\{1,2\}}D_{\{1,2,3,4\}} + D_{\{1,3\}}D_{\{1,2,3,4\}} + D_{\{1,4\}}D_{\{1,2,3,4\}} \\ &\quad + D_{\{2,3\}}D_{\{1,2,3,4\}} + D_{\{2,4\}}D_{\{1,2,3,4\}} + D_{\{3,4\}}D_{\{1,2,3,4\}}, \\ \delta_{3,4} &= D_{\{1,2,3\}}D_{\{1,2,3,4\}} + D_{\{1,2,4\}}D_{\{1,2,3,4\}} + D_{\{1,3,4\}}D_{\{1,2,3,4\}} + D_{\{2,3,4\}}D_{\{1,2,3,4\}}. \end{aligned}$$

From the argument given in section 6 we see that

$$F^*(D_I) = E_{\{1,2,3,4\}-I} \text{ when } |I| = 3, 4,$$

$$F^*(D_{\{i,4\}}) = a_i - E_0 - E_j - E_k,$$

$$F^*(D_{\{j,k\}}) = d_{j,k} - E_0 - E_i - E_4,$$

for $1 \leq i \leq 3$ and $j \neq k \in \{1, 2, 3\} - \{i\}$, from which we conclude that

$$F^*(\delta_{2,2}) = a_1d_{2,3} + a_2d_{1,3} + a_3d_{1,2} + 3E_0^2,$$

$$F^*(\delta_{2,3}) = 3(a_1a_2 + a_1a_3 + a_2a_3 + d_{1,2}d_{1,3}) + 3(4E_0 + E_1 + E_2 + E_3 + E_4)E_0,$$

$$F^*(\delta_{2,4}) = -3(2E_0 + E_1 + E_2 + E_3 + E_4)E_0,$$

$$F^*(\delta_{3,4}) = (E_1 + E_2 + E_3 + E_4)E_0.$$

If we substitute the expressions above into the relation (6), we get that

$$12(a_1d_{2,3} + a_2d_{1,3} + a_3d_{1,2} - a_1a_2 - a_1a_3 - a_2a_3 - d_{1,2}d_{1,3}) = 12(a_1b_{2,3} - b_{1,2}b_{1,3}) = 0.$$

The push-forward of the relation above via the blow-down map to C^3 gives the relation (2).

We next deal with the relation (3). Recall that $\pi : \overline{M}_{1,5} \rightarrow \overline{M}_{1,4}$ forgets the fifth marking. It induces a morphism from $M_{1,5}^{ct}$ to $M_{1,4}^{ct}$, which is denoted by the same letter by abuse of notation. We study the pull-back of the relation (6) along the morphism $\pi \circ F$, where $F : \overline{U}_4 \rightarrow M_{1,5}^{ct}$ is the morphism defined in the first section. It is easy to see that

$$\begin{aligned} (\pi \circ F)^*\delta_{2,2} &= d_{1,2}d_{3,4} + d_{1,3}d_{2,4} + d_{1,4}d_{2,3} + 3(E_0 + E_5)^2, \\ (\pi \circ F)^*\delta_{2,3} &= 12(E_0 + E_5)^2 + 3 \left(d_{1,2}d_{1,3} + d_{1,2}d_{1,4} + d_{1,3}d_{1,4} + d_{2,3}d_{2,4} + \sum_{i=1}^4 (E_0 + E_5)(E_i + E_{i,5}) \right), \\ (\pi \circ F)^*\delta_{2,4} &= -6(E_0 + E_5)^2 - \sum_{i \neq j \in \{1,2,3,4\}} d_{i,j}(E_{i,5} + E_{j,5}) - 3 \sum_{i=1}^4 E_0 E_i, \\ (\pi \circ F)^*\delta_{3,4} &= \sum_{i=1}^4 E_5 E_{i,5} + E_0(E_i + E_{i,5}). \end{aligned}$$

The substitution of the expressions above into relation (6) yields

$$12(b_{1,2}b_{3,4} + b_{1,3}b_{2,4} + b_{1,4}b_{2,3}) = 0,$$

whose push-forward to C^4 via the blow-down map is (3).

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